MATHEMATICAL DESCRIPTION OF THE HYDRODYNAMIC REGIMES OF AN ASYMPTOTIC MODEL FOR TWO-PHASE FLOW ARISING IN PFBC BOILERS

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Abstract: Two-phase systems where a dense phase of small particles is fluidized with a gas flow appear in many industrial applications, among which the fluidized bed combustors are probably the most important. A homogenization technique allows us to formulate the mathematical model in form of the compressible Navier-Stokes system type with some particularities: 1) the volumetric fraction of the dense phase (analogous to the density in the Navier-Stokes equations) may vanish, 2) the constitutive viscosity law may depend in a nonlinear form on this density, 3) the source term is nonlinear and coupled with state equations involving drag forces and hydrodynamic pressure, and 4) the state equation for the collision pressure of dense phase blows up for finite values of the density. We develop a rigorous theory for a special kind of solutions we call *stationary clouds*. Such solutions exist only under restrictions on the geometry of combustor and on the boundary conditions that usually meet in engineering applications. In return, these solutions have a stationary one-dimensional structure very simple and, from them, it is possible to reconstruct much of the dynamics of the whole system, responding to most of the practical issues of interest. Finally, we study the linear stability for the trivial solutions corresponding to uniform fluidized states injecting plane wave perturbations in our equations. Depending on the parameters of the equations of state describing the collisions between solid particles, hydrodynamic pressure, and the values of blowing boundary condition, we can draw detailed abacus separating stable regions of unstable regions where bubbles appear. Then, we use the dispersion relations of this multidimensional linearized model, combined with the stationary phase theorem, to approach the profiles and the evolution of the bubbles appearing in unstable regimes, and verify that the obtained results adjust to the observations.

Keywords: mathematical modeling, PFBC, two-phase flow, stability, bubbles

INTRODUCTION

The complete asymptotic mathematical model for a two-phase system where a dense phase of small particles is fluidized with a gas flow, can be written as a non-dimensional Navier-Stokes system (see [2], [3] and [4]):

Conservation Equations (1)

$$\left.\begin{array}{l} \left.\begin{array}{c} \partial_t\rho + div\ \mathrm{m} = 0 \\ \partial_t\mathrm{m} + \mathbf{div}\left(\frac{\mathrm{m}}{\rho}\otimes\mathrm{m}\right) + \nabla p = \frac{1}{\mathrm{Re}}\ \mathbf{div}\ \bar{\tau} - \nabla p_h - \frac{1}{Fr}\ \rho\ \mathbf{e}_d \end{array}\right\}\ \text{in } \Omega_+(t) \\ \quad\rho = 0, \quad \mathrm{m} = \mathbf{0} \qquad\qquad\qquad\qquad\qquad\ \ \text{in } \Omega_0(t) \\ \quad div\ \mathbf{M} = 0 \qquad\qquad\qquad\qquad\qquad\qquad\ \text{in } \Omega \end{array}\right\}\quad \forall t\in(0,T)$$

where ρ and **m** are the volume fraction, and momentum of the solid phase, p and p_h are the collision and hydrodynamic pressures, $Re>0$ and $Fr>0$ the usual non-dimensional Reynolds and Froude numbers, $\mathbf{e}_d$ the unit vector in the direction of gravity, and **M** the total momentum of both phases. The mass and momentum conservation equations (1) must be completed with the state equations for collision pressure p (describing interaction between solid particles) and hydrodynamic pressure p_h (in order to model interaction between phases), and also adequate constitutive laws for stresses $\boldsymbol{\tau}$ due to the viscosity of the solid phase, and a function Φ modeling the friction between phases:

Equations of State (2)

$$\left.\begin{aligned} \nabla p_h &= -\frac{1}{Fr}\frac{\rho}{\Phi}\left(\mathbf{M}-\frac{\mathbf{m}}{\rho}\right) \\ p &= p(\rho) \end{aligned}\right\} \quad \text{in } \Omega_+(t)$$

Constitutive Laws (3)

$$\left.\begin{aligned} \bar{\tau} &= \bar{\tau}\left(\rho, \frac{\mathbf{m}}{\rho}\right) \\ \Phi &= \Phi(\rho) \end{aligned}\right\} \quad \text{in } \Omega_+(t)$$

It is also necessary to complete the system with initial conditions for the momentum of dense phase and for the amount of solid mass and its initial distribution:

Initial Conditions (4)

$$\left.\begin{aligned} \rho(\mathbf{x},0) &= \rho^0(\mathbf{x}) \in [0,1] \ \text{ such that } \int_\Omega \rho^0(\mathbf{x})\, d\mathbf{x} = 1 \\ \mathbf{m}(\mathbf{x},0) &= \mathbf{m}^0(\mathbf{x}) \ \text{ such that } \mathbf{m}^0 = \mathbf{0} \ \text{ in } \Omega_0(0) \end{aligned}\right\} \quad \text{in } \overline{\Omega}$$

and with adequate boundary conditions on all the boundary δΩ and on the “input” boundary Γ_{+0} and “exit” boundary Γ_{-}:

Boundary Conditions (5)

$$\left.\begin{aligned} \mathbf{m}\cdot\boldsymbol{\nu} &= 0 \ \text{ if } \mathrm{Re} = +\infty \\ \mathbf{m} &= \mathbf{0} \ \text{ if } \mathrm{Re} \in \Re_+ \end{aligned}\right\} \quad \text{on } \partial\Omega\times(0,T) \qquad \begin{aligned} \mathbf{M}\cdot\boldsymbol{\nu} &= -M_0 \quad \text{on } \Gamma_{+0} \\ p_h &= 0 \quad \text{on } \Gamma_{-} \end{aligned}$$

being **ν** the outer unit normal on the boundary. Finally, we must add a condition on the free boundary Γ that separates the zone where we have solid particles from the area in which there is only gas phase:

Free Boundary Condition (6)

$$\mathbf{m}\cdot\boldsymbol{\nu} = 0 \ \text{ and } \bar{\boldsymbol{\sigma}}\,\boldsymbol{\nu} = \mathbf{0} \quad \text{on } \Gamma(t), \quad \forall t \in (0,T)$$

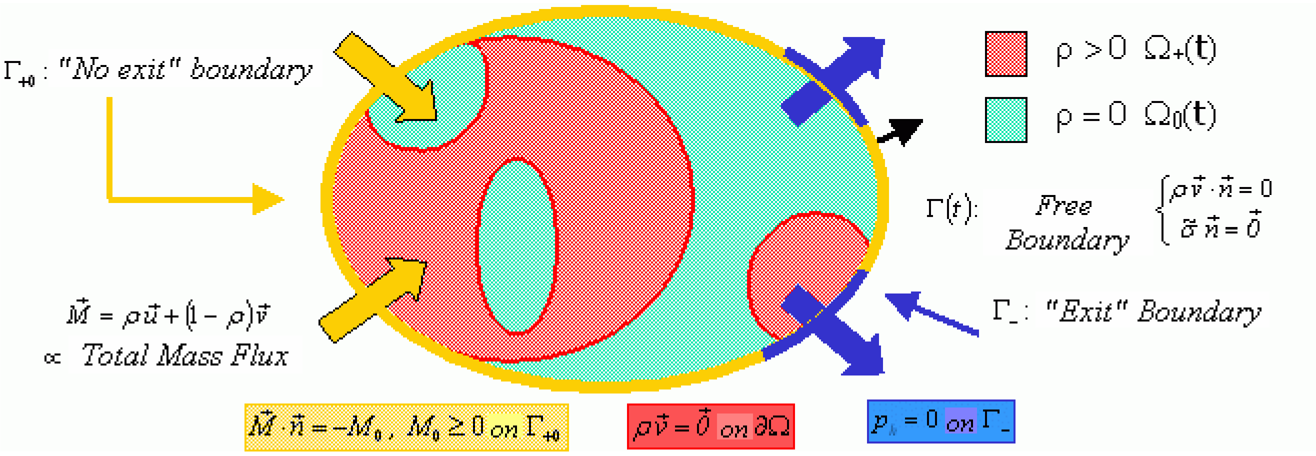


Fig. 1: Boundary Conditions

STATIONARY CLOUDS AND STATES OF UNIFORM FLUIDIZATION

Our aim is the rigorous (mathematical) study of two key aspects of the model (1) to (6). First, what we might call the *Rigorous Engineering Theory of Fluid Beds*. This engineering theory is based on the study of a particular class of simple solutions to the whole system (1) to (6), which we have called *solutions of stationary cloud kind.* From this type of particular simple solutions, anyone can rebuild the qualitative dynamics of the whole system, responding to the majority, or at least many of the practical issues of interest in engineering applications. We say that this theory is rigorous, because it gives a precise answer, obtained only from the equations of the model, to the key issue of under what conditions it is justified to consider such simple solutions, and what are its essential properties. For a summary of this theory, see [5] and [6]. We define the ***stationary clouds*** as solutions of (1) to (6) such that $\rho \geq 0$ and **m** = **0**. Such solutions do not always exist. We can prove that these engineering solutions are justified only by assuming important geometric constraints for the domain and its boundary, which meet or do much about, in most industrial designs.

The precise conditions assuring *existence of stationary clouds* are the following:

- The size of the domain (combustion chamber of PFBC boilers) is large enough: $|\Omega| > 1$.
- Compatibility between the domain geometry and the *blowing boundary condition* (see Fig. 2):

$$\Gamma_{+0} = \{\vec{x} \in \partial\Omega : \vec{g}\cdot\vec{n} \geq 0\} \qquad \textit{There is a constant } M \geq 0 \textit{ such that :}$$
$$\Gamma_{-} = \{\vec{x} \in \partial\Omega : \vec{g}\cdot\vec{n} < 0\} \qquad M_0 = M_0(\vec{x}) = -M\left(\vec{e}_d \cdot n(\vec{x})\right)\ \forall \vec{x} \in \Gamma_{+0}$$

- An additional geometrical restriction (see Fig. 3): Ω *is connected and bounded in the directions* x_i $i=1,\ldots,d-1$, *and simply connected in these directions*.

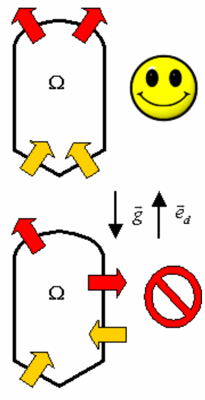


Fig.2: Necessary compatibility between the domain geometry and the blowing boundary condition

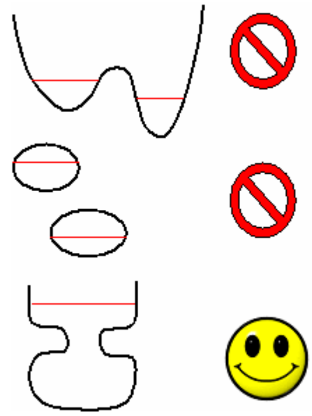


Fig. 3: Necessary properties about domain connection

In these conditions, we can proof the **existence of such stationary clouds** characterized by having one-dimensional structure in which $x = x_d$, and verifying:

$\vec{M} = M\,\vec{e}_d\ \left(M \geq 0 \textit{ constant}\right) \textit{and } \rho = \rho(x) \geq 0 \textit{ such that :}$

$$\frac{dp}{dx} = -\frac{\rho}{Fr}\left(1 - \frac{M}{\phi}\right) \textit{ when } \rho > 0 \tag{7}$$

with the state equation p=p(ρ) increasing, the constitutive law φ=φ(ρ) decreasing, both C^1 *functions, and the normalization condition:* $\int_\Omega \rho\, dx = 1$ *(total mass fixed).*

The solutions to the non-linear problem (7) may or may not have free boundary (which separates the area with dense phase particles, or *bed*, from the area free of them, or *freeboard*), depending on the nature of the domain (combustion chamber) (bounded or unbounded) and of his size. The existence of a free boundary (and his position) depends also on the total mass of particles and on the magnitude of the boundary condition (blowing gas). In addition, we proof that the regularity of the solution depends essentially on the properties of the equation of state for the pressure of collision between solid particles. *The main result states that (7) has a unique weak solution* $\rho \in C^0(\Omega) \cap C^1(\rho > 0)$. *This solution is a classical solution* $\rho \in C^1(\Omega)$, *if and only if: (a) Or* Ω *is bounded and there is not free boundary, (b) Or* Ω *is unbounded and* $\lim_{\rho \to 0_+} \rho / p'(\rho) = 0$ (8)

In particular, in the very important case of a equation of state for the dense phase of the form:

$$p(\rho)=\alpha^2\rho^{\gamma_0}L(\rho/\rho_*),\ \gamma_0>1,\ \text{with}\ \lim_{\varepsilon\to 0_+}L_\varepsilon(z)=1,\ \lim_{\varepsilon\to 0_+}\frac{L'_\varepsilon(z)}{L_\varepsilon(z)}=1\ \textit{uniformly}\ \forall z\in[0,1) \quad (9)$$

The condition (8) corresponds to $1<\gamma_0<2$. For $\gamma_0\geq 2$ the global solution is only a weak solution.

Behavior of the stationary clouds: Profiles of the solutions

Especially, we have focused our attention on studying the *dependence of the position of the free boundary in function of the magnitude of the blowing boundary condition*. This allows us, for example, to design practical policies to minimize erosion problems in the reactor.

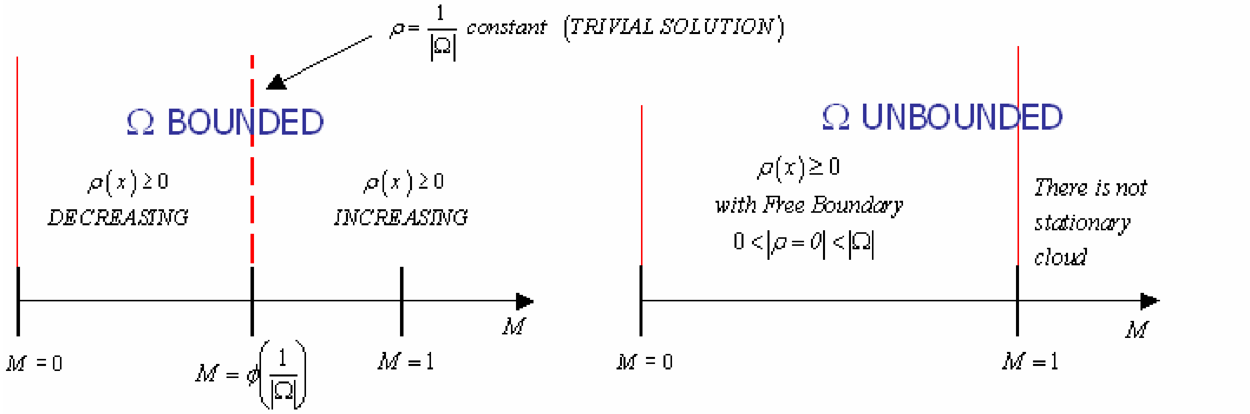


Fig. 3: Qualitative behavior in function of the boundary condition (blowing Mach M)

We can proof theorems (see [6]) about the global behavior of the profiles of solutions to the stationary clouds problem (7) (see Fig,3). More precisely, calling *h=h(M)* for the position of the free boundary which corresponds to the *bed height* in engineering language, we can consider the following cases:

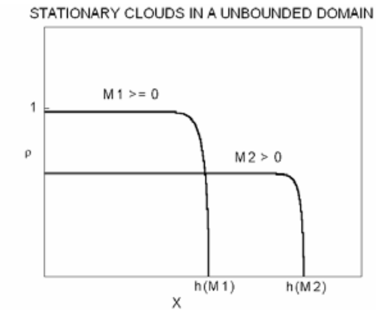


Fig.4: Stationary clouds in a unbounded domain

1. *Unbounded domain case (open or very large unrealistic reactor)*: In this case, there is always a free boundary so that its height *h(M)* verifies:

   $$h:[0,1)\to\mathbb{R}_+\ \textit{increasing and s.t.}\ \lim_{M\to 1\_}h(M)=+\infty$$

   and *M=1* appears as the blowing boundary condition for which the fluid bed disappears (ρ=0). For this reason, *M=1* is called *flying Mach*. See Fig.4.

2. *Bounded domain. Case 1: short reactors.* In this case there is not a free boundary for the blowing boundary condition less than a *critical Mach M** verifying: $M^*>\phi(1/|\Omega|)>0$.For *M>M**, it appears a free boundary whose position *h(M)* is bounded by the size of the reactor:

   $$h:(M^*,+\infty)\to\mathbb{R}_+\ \textit{strictly increasing s.t.}\ \lim_{M\to M^*}h(M)=0_+\ \textit{and}\ h(M)<|\Omega|-1.$$

   This case only appears if the reactor is sufficiently short, i.e. if his height *L<L**, where the *critical height L** is given by: $L^*=1/\phi(M^*)$.

3. *Bounded domain. Case 2: long reactors (the most usual case for PFBC boilers).* In this case there is always an initial free boundary, leaving a *free board* in the reactor. Its height grows with the blowing boundary condition up to a first critical Mach M_0 such that:

   $$0<M_0<\phi(1/|\Omega|)\ \textit{and}\ h_0:[0,M_0)\to\mathbb{R}_+\ \textit{strictly increasing and s.t.}\ \lim_{M\to M_0}h_0(M)=|\Omega|$$

   For $M_0<M<M_1$, being M_1 a second critical Mach verifying $M_1>\phi(1/|\Omega|)$, the bed completely fills the reactor without free boundary. For $M>M_1$, a free boundary appears again, whose position is such that: $h_1:(M_1,+\infty)\to\mathbb{R}_+$ *strictly increasing s.t.* $\lim\limits_{M\to M_{1+}} h_1(M)=0_+$ *and* $h_1(M)<|\Omega|-1$, leaving a particles free area in the bottom of the reactor.

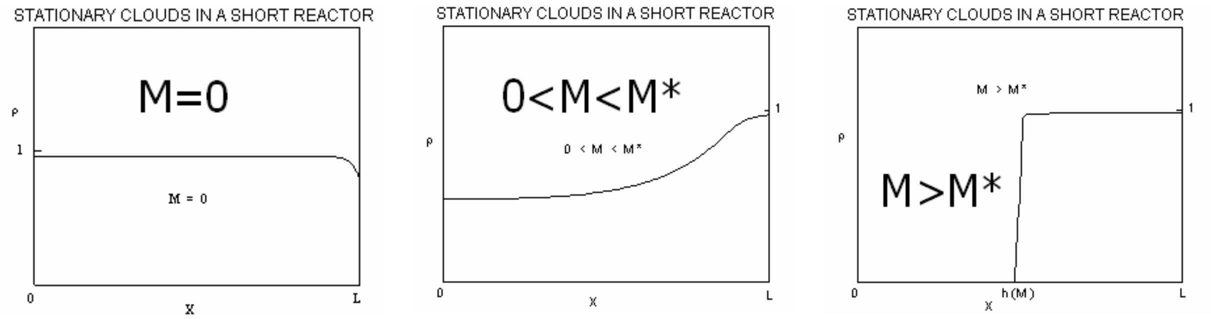


Fig.5: Stationary clouds in a bounded and short reactor

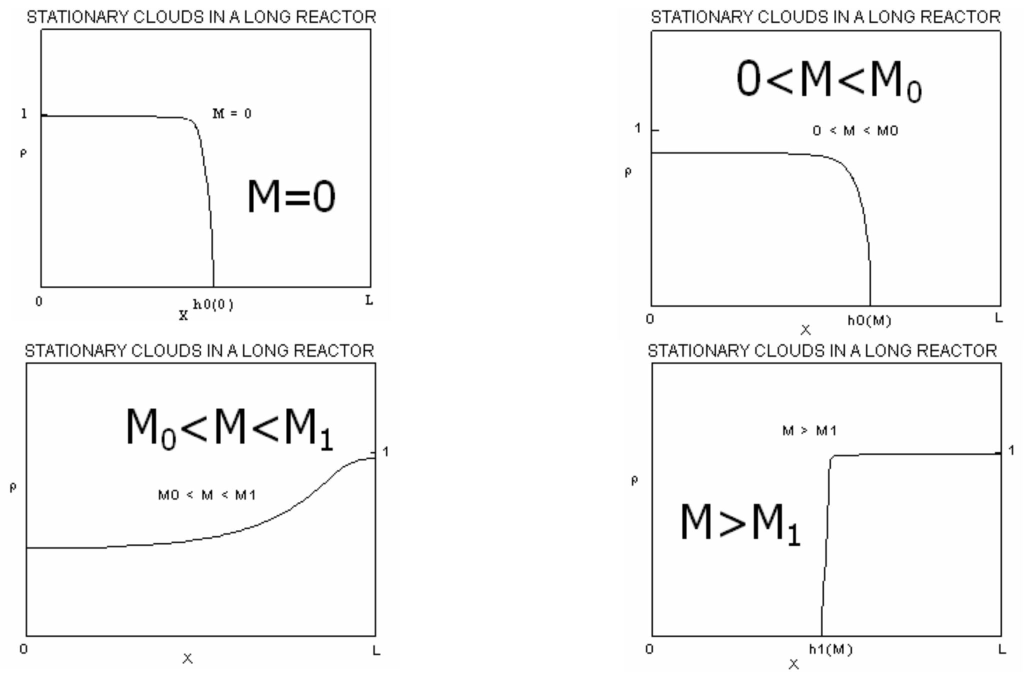


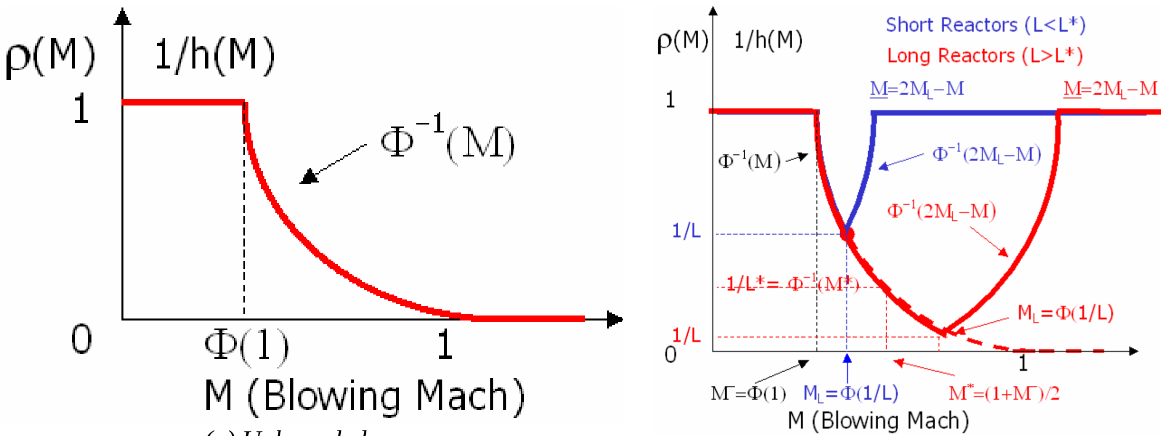
Fig.6: Stationary clouds in a bounded and long reactor

Engineering theory of uniform fluidization states

Another key issue addressed in our study was which we have called *Engineering theory of fluidized beds*, based on *trivial solutions* of (7), which are nothing other than the uniform distributions of volume fraction of dense phase, widely used in Engineering because its simplicity. These simple distributions, are really a solution to the problem (7) only in the case of a bounded domain and for certain privileged values of the blowing boundary condition. In fact, the *states of uniform fluidization* can also be defined as a limit of the *stationary solutions* (7) when $Fr\to+\infty$. These states are only determined by the properties of the law of friction between the solid and gas phases Φ, but they not depend, however, of the equation of state for the pressure of collision between particles. Then, we can easily proof that, *in the case of an unbounded, or very large reactor*, the *states of uniform fluidization* satisfy:

$$\rho(M)=\left\{\begin{array}{lr} 1 & if\ M\in\left[0,\phi(1)\right] \\ \overline{\rho}\in(0,1),\ unique\ solution\ of\ \ \phi(\rho)=M & if\ M\in\left(\phi(1),1\right] \\ 0 & if\qquad M>1 \end{array}\right\}\ ;\ h(M)=\frac{1}{\rho(M)}$$

where $M_-=\phi(1)$ is known as the *Mach of minimum fluidization*. In the case of bounded domains, the theory becomes much more complex (see Fig. 7, (b)).



(a) Unbounded case *(b) Bounded case*

Fig.7: Engineering theory of fluidized beds

STABILITY OF TRIVIAL SOLUTIONS AND BUBBLES

The interest of the study conducted here is to highlight that, depending mainly on the properties of the equation of state for the pressure and thus on the regularity of the *stationary clouds*, may appear regimes in which such simple solutions are linearly unstable. This question is not merely formal, since the existence of unstable regimes lies the interest of the industrial use of fluid bed systems, thus contributing to the intimate mixing of particles in the dense phase, and promoting chemical reactions to be seized. To study the *linear stability* of the *trivial solution*, we introduce small perturbations. Then, the following linear equations for small perturbations holds:

$$\left.\begin{array}{c}\partial_t\left(\delta\rho\right)+div\ \left(\delta\mathbf{m}\right)=0\\ \partial_t\left(\delta\mathbf{m}\right)+p'\left(\overline{\rho}\right)\nabla\left(\delta\rho\right)=\frac{1}{Re}\ \mathbf{div}\ \left(\delta\tau\right)-\nabla\left(\delta p_h\right)-\frac{1}{Fr}\ \left(\delta\rho\right)\ \mathbf{e}_d\\ div\ \left(\delta\mathbf{M}\right)=0\end{array}\right\}$$

$$\nabla\left(\delta p_h\right)=-\frac{1}{Fr\ \Phi\left(\overline{\rho}\right)}\ \left(\overline{\rho}\ \left(\delta\mathbf{M}\right)-\left(\delta\mathbf{m}\right)+\left(1-\overline{\rho}\ \frac{\Phi'\left(\overline{\rho}\right)}{\Phi\left(\overline{\rho}\right)}\right)\ \Phi\left(\overline{\rho}\right)\ \left(\delta\rho\right)\ \mathbf{e}_d\right)$$

Then, looking for solutions of plane wave kind:

$$\delta\varphi=\delta\varphi_0\ exp\ i\left(\vec{k}\cdot\vec{x}-\omega t\right)\ exp\left(\Omega t\right)\quad with\quad \left|\vec{k}\right|=2\pi/\lambda$$

we can proof that the plane wave are linearly stable if and only if:

$$\Omega\geq 0\ \Leftrightarrow\ M_g=-\frac{\overline{\rho}\,\phi'\left(\overline{\rho}\right)}{\sqrt{p'\left(\overline{\rho}\right)}}\leq 1\ \Leftrightarrow\ M_p^2=\left(\frac{\omega}{\left|\vec{k}\right|}\right)^2\frac{1}{p'\left(\overline{\rho}\right)}\leq 1 \tag{10}$$

being M_g and M_p the Mach number for the perturbations of the gaseous phase and of particle phase, respectively. For the particular, but important case in practice, of a state equation for collision pressure as (9) and a law of friction between both phases of the form:

$$\phi\left(\rho\right)=\left(1-\rho_*\,\rho\right)^{m+1}\quad with\quad \rho_*\in\left(0,1\right)\ \ and\ \ m\geq 0$$

we obtain the picture of the region where the trivial solutions are linearly stable in the plane Mach of blowing (M) versus pressure coefficient (α in (9)). As it is shown in Fig. 8, for a particular value $\gamma_0=3$ of the exponent of the equation of state for collision pressure between particles, we obtain a region of absolute stability when a high pressure happens. If $\gamma_0>3$ the absolute stability for high pressure is asymptotic. For $\gamma_0<3$ we also obtain a region of marginal stability for high blowing Mach (not shown in Fig. 8). Finally, we have formally proved that the mechanism by showing the instability is manifested in the formation of bubbles within the dense phase. These bubbles, or areas with very few particles, are increasing in size, and they travel through the dense phase at a constant speed given by M_p in (10). This can be shown by calculating the asymptotic form of disturbances for small values of the wavelength λ:

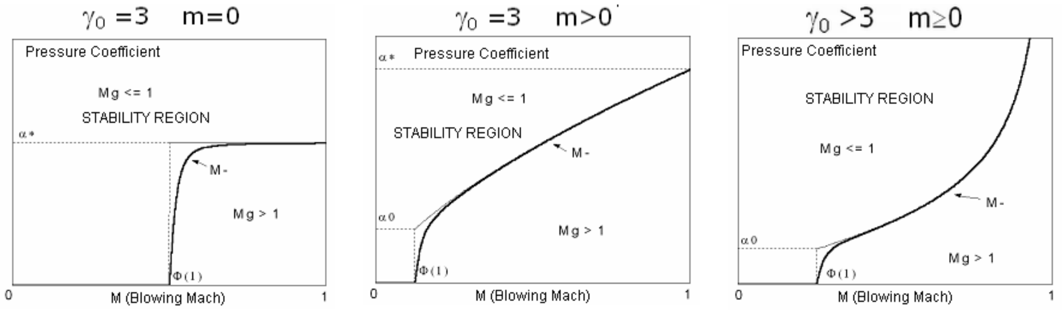


Fig.8: Linear stability regions for trivial solutions

$$\delta\rho(\mathbf{x},t;\lambda) = \frac{1}{2\,\pi\,|\mathbf{x}|}\left(sen\ \varphi_+\ e^{\Omega_+\ t} + sen\ \varphi_-\ e^{\Omega_-\ t}\right) + O\left(\left(\frac{\lambda}{|\mathbf{x}|}\right)^2\right)$$

$$\Omega_\pm = \Omega\left(\pm\frac{2\,\pi}{\lambda}\,\frac{\mathbf{x}}{|\mathbf{x}|}\right),\quad \omega_\pm = \omega\left(\pm\frac{2\,\pi}{\lambda}\,\frac{\mathbf{x}}{|\mathbf{x}|}\right),\quad \varphi_\pm = \frac{2\,\pi}{\lambda}\,|\mathbf{x}| \mp \omega_\pm\, t$$

Then, the bubbles appears from the interaction of damping backward waves and amplified forward waves. The results obtained are substantially consistent with experimental observations described in the literature.

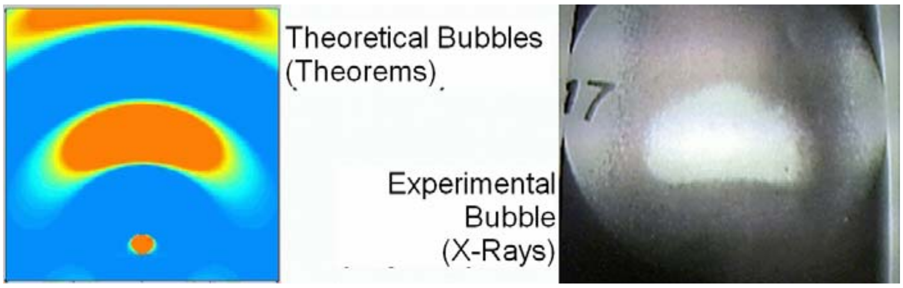


Fig. 9: Comparison between theoretical mechanism for bubbles and a experimental bubble (this last from [1])

CONCLUSIONS

From an asymptotic model of two-phase flow we have been identified and studied the stationary solutions that recover the behavior of clouds of particles in gas flows. We have derived, as a limit of these solutions, when the Froude number tends to infinity, a rigorous engineering theory about uniform fluidization states arising in both short and long reactors. Finally, we have studied the linear stability of these uniform fluidization states, showing the conditions under which bubbles are produced in the fluidized bed. These conditions depend on the equation of state of the solid phase and also on the law of friction between the two phases, but they are viscosity independent. The analysis shows that the bubbles are generated by small perturbations whose propagation speed is higher than the speed of the collision waves between particles.

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